

# ASYMPTOTIC CONDITIONAL DISTRIBUTION OF EXCEEDANCE COUNTS: FRAGILITY INDEX WITH DIFFERENT MARGINS

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**ABSTRACT.** Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector, whose components are not necessarily independent nor are they required to have identical distribution functions  $F_1, \dots, F_d$ . Denote by  $N_s$  the number of exceedances among  $X_1, \dots, X_d$  above a high threshold  $s$ . The fragility index, defined by  $FI = \lim_{s \nearrow} E(N_s \mid N_s > 0)$  if this limit exists, measures the asymptotic stability of the stochastic system  $\mathbf{X}$  as the threshold increases. The system is called stable if  $FI = 1$  and fragile otherwise. In this paper we show that the asymptotic conditional distribution of exceedance counts (ACDEC)  $p_k = \lim_{s \nearrow} P(N_s = k \mid N_s > 0)$ ,  $1 \leq k \leq d$ , exists, if the copula of  $\mathbf{X}$  is in the domain of attraction of a multivariate extreme value distribution, and if  $\lim_{s \nearrow} (1 - F_i(s))/(1 - F_k(s)) = \gamma_i \in [0, \infty)$  exists for  $1 \leq i \leq d$  and some  $\kappa \in \{1, \dots, d\}$ . This enables the computation of the FI corresponding to  $\mathbf{X}$  and of the extended FI as well as of the asymptotic distribution of the exceedance cluster length also in that case, where the components of  $\mathbf{X}$  are not identically distributed.

## 1. INTRODUCTION

Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector (rv), whose components are identically distributed but not necessarily independent. The number of exceedances among  $X_1, \dots, X_d$  above the threshold  $s$  is denoted by  $N_s := \sum_{i=1}^d 1_{(s, \infty)}(X_i)$ . The fragility index (FI) corresponding to  $\mathbf{X}$  is the asymptotic conditional expected number of exceedances, given that there is at least one exceedance, i.e.,  $FI = \lim_{s \nearrow} E(N_s \mid N_s > 0)$ . The FI was introduced in Geluk et al. (2007) to measure the stability of the stochastic system  $\{X_1, \dots, X_d\}$ . The system is called *stable* if  $FI = 1$ , otherwise it is called *fragile*.

In the 2-dimensional case, the FI is directly linked to the upper tail dependence coefficient  $\lambda^{up} := \lim_{t \downarrow 0} P(X_2 > F_2^{-1}(1-t) \mid X_1 > F_1^{-1}(1-t))$ , which goes back to Geffroy (1958, 1959) and Sibuya (1960). We have  $FI = 2/(2 - \lambda^{up})$ , provided the df  $F_1, F_2$  of  $X_1, X_2$  are continuous and  $\lambda^{up}$  exists. In contrast to the upper tail dependence coefficient, the FI presents a measure for tail dependence in an arbitrary dimensions.

In Falk and Tichy (2010) the asymptotic conditional distribution  $p_k := \lim_{s \nearrow} P(N_s = k \mid N_s > 0)$  of the number of exceedances was investigated, given that there is at least one exceedance,  $1 \leq k \leq d$ .

It turned out that this *asymptotic conditional distribution of exceedance counts* (ACDEC) exists, if the copula  $C$  corresponding to  $\mathbf{X}$  is in the domain of attraction of a (multivariate) extreme value distribution (EVD)  $G$ , denoted by  $C \in D(G)$ , i.e.  $C^n \left( \left(1 + \frac{x_1}{n}, \dots, 1 + \frac{x_d}{n}\right) \right) \rightarrow_{n \rightarrow \infty} G(\mathbf{x})$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ .

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In this paper we investigate the ACDEC, dropping the assumption that the margins  $X_i$ ,  $1 \leq i \leq d$ , are identically distributed. This will be done in Section 2. If the ACDEC exists then the FI exists and we have in particular  $FI = \sum_{k=1}^d kp_k$ . In Section 3 we will compute the FI under quite general conditions on  $\mathbf{X}$ .

The extended fragility index  $FI(m)$  is the extension of the  $FI = FI(1)$  through the condition that there are at least  $m \geq 1$  exceedances, i.e.,

$$FI(m) = \lim_{s \nearrow} E(N_s \mid N_s \geq m) = \frac{\sum_{k=m}^d kp_k}{\sum_{k=m}^d p_k},$$

if the ACDEC exists. But now we encounter the problem that the denominator in the definition of  $FI(m)$  may vanish:  $\sum_{k=m}^d p_k = 0$ . In Section 4 we will establish a characterization of  $\sum_{k=m}^d p_k = 0$  in terms of tools from multivariate extreme value theory.

The total number of sequential time points at which a stochastic process exceeds a high threshold is an *exceedance cluster length*. The asymptotic distribution as the threshold increases of the remaining exceedance cluster length, conditional on the assumption that there is an exceedance at index  $\kappa \in \{1, \dots, d\}$ , will be computed for  $\mathbf{X} = (X_1, \dots, X_d)$  in Section 5. It turns out that this can be expressed in terms of the minimum of a *generator* of the  $D$ -norm, cf equation (4.2).

## 2. ACDEC

By Sklar's Theorem (see, for example, (Nelsen, 2006, Theorem 2.10.9)) we can assume the representation  $(X_1, \dots, X_d) = (F_1^{-1}(U_1), \dots, F_d^{-1}(U_d))$ , where  $F_i$  is the (univariate) distribution function (df) of  $X_i$ ,  $1 \leq i \leq d$ , and the rv  $\mathbf{U} = (U_1, \dots, U_d)$  follows a copula on  $\mathbb{R}^d$ , i.e., each  $U_i$  is uniformly on  $(0, 1)$  distributed,  $1 \leq i \leq d$ . By  $F^{-1}(q) := \inf \{t \in \mathbb{R} : F(t) \geq q\}$ ,  $q \in (0, 1)$ , we denote the generalized inverse of a df  $F$ .

The following condition is crucial for the present paper. It substitutes the condition of equal margins  $F_1 = \dots = F_d$  in Falk and Tichy (2010). By  $\omega(F) := \sup \{F^{-1}(q) : q \in (0, 1)\} = \sup \{t \in \mathbb{R} : F(t) < 1\}$  we denote the upper endpoint of a df  $F$ .

We require throughout the existence of an index  $\kappa \in \{1, \dots, d\}$  with  $\omega(F_\kappa) =: \omega^*$ , such that

$$(C) \quad \lim_{s \uparrow \omega^*} \frac{1 - F_i(s)}{1 - F_\kappa(s)} = \gamma_i \in [0, \infty), \quad 1 \leq i \leq d.$$

Note that condition (C) implies  $\omega(F_i) \leq \omega^*$  for each  $i$ , since otherwise we would get  $\gamma_i = \infty$ , which is excluded. We, thus, have  $\omega^* = \max_{i \leq d} \omega(F_i)$ .

The following result is taken from Aulbach et al. (2011). By  $\mathbf{e}_i$  we denote the  $i$ -th unit vector in  $\mathbb{R}^d$ ,  $1 \leq i \leq d$ ; all operations on vectors such as  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$  are meant componentwise.

**Proposition 2.1.** *An arbitrary copula  $C$  on  $\mathbb{R}^d$  is in the domain of attraction of an EVD  $G$  if and only if there exists a norm  $\|\cdot\|_D$  on  $\mathbb{R}^d$  with  $\|\mathbf{e}_i\|_D = 1$ ,  $1 \leq i \leq d$ , such that*

$$C(\mathbf{y}) = 1 - \|\mathbf{y} - \mathbf{1}\|_D + o(\|\mathbf{y} - \mathbf{1}\|_D),$$

*uniformly for  $\mathbf{y} \in [0, 1]^d$ . In this case  $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_D)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ .*

The following result is an immediate consequence of Proposition 2.1 and the equivalence  $F^{-1}(q) \leq t \iff q \leq F(t)$ ,  $q \in (0, 1)$ ,  $t \in \mathbb{R}$ , which holds for an arbitrary df  $F$ .

**Corollary 2.2.** *Suppose that the copula  $C$  corresponding to the rv  $\mathbf{X}$  is in the domain of attraction of an EVD  $G$  and that condition (C) is satisfied. Then there exists a norm  $\|\cdot\|_D$  on  $\mathbb{R}^d$  with  $\|\mathbf{e}_i\|_D = 1$ ,  $1 \leq i \leq d$ , such that for any nonempty index set  $K \subset \{1, \dots, d\}$*

$$P(X_k \leq s, k \in K) = 1 - (1 - F_\kappa(s)) \left\| \sum_{k \in K} \gamma_k \mathbf{e}_k \right\|_D + o(1 - F_\kappa(s))$$

as  $s \uparrow \omega^*$ .

The following result provides the asymptotic unconditional distribution of exceedance counts.

**Lemma 2.3.** *Under the conditions of Corollary 2.2 we obtain with  $c := 1 - F_\kappa(s)$*

$$\begin{aligned} a_k &:= \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{c} \\ &= \sum_{0 \leq j \leq k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{\emptyset \neq T \subset \{1, \dots, d\} \\ |T|=d-j}} \left\| \sum_{i \in T} \gamma_i \mathbf{e}_i \right\|_D \end{aligned}$$

for  $1 \leq k \leq d$ , and

$$a_0 := \lim_{s \uparrow \omega^*} \frac{1 - P(N_s = 0)}{c} = \left\| \sum_{j=1}^d \gamma_j \mathbf{e}_j \right\|_D.$$

*Proof.* Corollary 2.2 implies

$$P(N_s = 0) = 1 - c \left\| \sum_{j=1}^d \gamma_j \mathbf{e}_j \right\|_D + o(c),$$

for  $s \uparrow \omega^*$ .

From the additivity formula, Corollary 2.2 and the equality  $\sum_{\emptyset \neq T \subset S} (-1)^{|T|+1} = 1$  for any nonempty subset  $S \subset \{1, \dots, d\}$ , we obtain for  $1 \leq k \leq d$  as  $s \uparrow \omega^*$

$$\begin{aligned} &P(N_s = k) \\ &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} P(X_i > s, i \in S, X_j \leq s, j \in S^c) \\ &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} P(X_i > s, i \in S \mid X_j \leq s, j \in S^c) P(X_j \leq s, j \in S^c) \\ &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} \left( 1 - \sum_{\emptyset \neq T \subset S} (-1)^{|T|+1} P(X_i \leq s, i \in T \mid X_j \leq s, j \in S^c) \right) \\ &\quad \times P(X_j \leq s, j \in S^c) \\ &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} \left( P(X_j \leq s, j \in S^c) - \sum_{\emptyset \neq T \subset S} (-1)^{|T|+1} P(X_i \leq s, i \in T \cup S^c) \right) \\ &= \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} \left( 1 - c \left\| \sum_{j \in S^c} \gamma_j \mathbf{e}_j \right\|_D - \sum_{\emptyset \neq T \subset S} (-1)^{|T|+1} \left( 1 - c \left\| \sum_{j \in T \cup S^c} \gamma_j \mathbf{e}_j \right\|_D \right) \right) \\ &\quad + o(c) \end{aligned}$$

$$= c \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} \sum_{T \subset S} (-1)^{|T|+1} \left\| \sum_{j \in T \cup S^c} \gamma_j \mathbf{e}_j \right\|_D + o(c).$$

With a suitable index transformation we get

$$\begin{aligned} P(N_s = k) &= c \sum_{\substack{S \subset \{1, \dots, d\} \\ |S|=k}} \sum_{0 \leq r \leq |S|} (-1)^{r+1} \sum_{\substack{K \subset S \\ |K|=r}} \left\| \sum_{\substack{i \in K \cup S^c \\ |T|=r+d-k}} \gamma_i \mathbf{e}_i \right\|_D + o(c) \\ &= c \sum_{0 \leq j \leq k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subset \{1, \dots, d\} \\ |T|=d-j}} \left\| \sum_{j \in T} \gamma_j \mathbf{e}_j \right\|_D + o(c), \end{aligned}$$

which completes the proof of Lemma 2.3.  $\square$

Note that  $a_0 > 0$  as  $\gamma_k = 1$  and that  $a_k \geq 0$ ,  $1 \leq k \leq d$ , in Lemma 2.3. The following main result of this section is, therefore, an immediate consequence of Lemma 2.3. It provides the ACDEC also in the case, where the components  $X_i$  of the rv  $\mathbf{X} = (X_1, \dots, X_d)$  are not identically distributed.

**Theorem 2.4** (ACDEC). *Under the conditions of Corollary 2.2 we have that the limits*

$$p_k := \lim_{s \uparrow \omega^*} P(N_s = k \mid N_s > 0) = \frac{a_k}{a_0}, \quad 1 \leq k \leq d,$$

*exist and that they define a probability distribution on  $\{1, \dots, d\}$ .*

For the usual  $\lambda$ -norm  $\|\mathbf{x}\|_\lambda = \left( \sum_{1 \leq i \leq d} |x_i|^\lambda \right)^{1/\lambda}$ ,  $\mathbf{x} \in \mathbb{R}^d$ ,  $\lambda \in [1, \infty)$ , we obtain, for example,  $a_0 = \left( \sum_{1 \leq i \leq d} \gamma_i^\lambda \right)^{1/\lambda}$  and

$$a_k = \sum_{0 \leq j \leq k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{\emptyset \neq T \subset \{1, \dots, d\} \\ |T|=d-j}} \left( \sum_{i \in T} \gamma_i^\lambda \right)^{1/\lambda}, \quad 2 \leq k \leq d.$$

For  $\lambda = 1$ , which is the case of independent margins of  $G$ , we obtain in particular  $a_0 = \sum_{1 \leq i \leq d} \gamma_i = a_1$ ,  $a_k = 0$ ,  $2 \leq k \leq d$ , and, thus,  $p_1 = 1$ ,  $p_k = 0$ ,  $2 \leq k \leq d$ .

### 3. THE FRAGILITY INDEX

The following theorem is the main result of this section.

**Theorem 3.1.** *Under the conditions of Corollary 2.2 we have*

$$FI = \frac{\sum_{i=1}^d \gamma_i}{\left\| \sum_{i=1}^d \gamma_i \mathbf{e}_i \right\|_D} \in [1, d].$$

*Proof.* We have

$$\begin{aligned} E(N_s \mid N_s > 0) &= \sum_{i=1}^d E(1_{(s, \infty)}(X_i) \mid N_s > 0) \\ &= \sum_{i=1}^d \frac{P(X_i > s)}{1 - P(N_s = 0)} \\ &= \sum_{i=1}^d \frac{1 - F_i(s)}{1 - F_\kappa(s)} \frac{1 - F_\kappa(s)}{1 - P(N_s = 0)} \end{aligned}$$

$$\rightarrow_{s \rightarrow \infty} \frac{\sum_{i=1}^d \gamma_i}{\left\| \sum_{i=1}^d \gamma_i \mathbf{e}_i \right\|_D}.$$

by Lemma 2.3 and condition (C).  $\square$

It is well known that an arbitrary  $D$ -norm satisfies the inequality  $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_D \leq \|\mathbf{x}\|_1$ ,  $\mathbf{x} \geq \mathbf{0} \in \mathbb{R}^d$ ; see, for example (Falk et al., 2010, (4.37)). The range of the FI in Theorem 3.1 is, consequently,  $[1, d]$ .

Suppose that  $\gamma_i > 0$ ,  $1 \leq i \leq d$ . Then it follows from Takahashi (1988) that

$$\left\| \sum_{i=1}^d \gamma_i \mathbf{e}_i \right\|_D = \sum_{i=1}^d \gamma_i \iff \|\cdot\|_D = \|\cdot\|_1,$$

where  $\|\cdot\|_D = \|\cdot\|_1$  is the case of independence of the margins of  $G$ . We, thus, obtain in case  $\gamma_i > 0$ ,  $1 \leq i \leq d$ ,

$$FI = 1 \iff \|\cdot\|_D = \|\cdot\|_1 \iff \text{independence of the margins of } G.$$

In case of complete dependence of  $G$ , i.e., if  $\|\mathbf{x}\|_D = \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq d} |x_i|$ , we obtain for general  $\gamma_i \geq 0$  that  $FI = \sum_{i=1}^d \gamma_i / \max_{1 \leq i \leq d} \gamma_i$ .

*Example 3.2* (Weighted Pareto). Let  $Y_1, \dots, Y_m$  be independent and identically Pareto distributed rv with parameter  $\alpha > 0$ . Put  $X_i := \sum_{j=1}^m \lambda_{ij} Y_j$ ,  $1 \leq i \leq d$ , where the weights  $\lambda_{ij}$  are nonnegative and satisfy  $\sum_{j=1}^m \lambda_{ij}^\alpha = 1$ ,  $1 \leq i \leq d$ .

The df of the rv  $\mathbf{X} = (X_1, \dots, X_d)$  is in the domain of attraction of the EVD

$$G^*(\mathbf{s}) = \exp \left( - \sum_{j=1}^m \max_{i \leq d} \left( \frac{\lambda_{ij}}{s_i} \right)^\alpha \right), \quad \mathbf{s} = (s_1, \dots, s_d) > \mathbf{0},$$

with standard Fréchet margins  $G_k(\mathbf{s}) = \exp(-s^{-\alpha})$ ,  $s > 0$ ,  $1 \leq k \leq d$ . This can be seen by proving that for  $\mathbf{s} > \mathbf{0} \in \mathbb{R}^d$

$$P \left( \sum_{j=1}^m \lambda_{ij} Y_j \leq n^{1/\alpha} s_i, 1 \leq i \leq d \right) = 1 - \frac{1}{n} \left( \sum_{j=1}^m \max_{i \leq d} \left( \frac{\lambda_{ij}}{s_i} \right)^\alpha + o(1) \right),$$

which follows from tedious but elementary computations, using conditioning on  $Y_j = y_j$ ,  $j = 2, \dots, m$ .

We, thus, obtain that the copula pertaining to  $\mathbf{X}$  is in the domain of attraction of  $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_D)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ , where  $\|\mathbf{x}\|_D := \sum_{j=1}^m (\max_{i \leq d} (\lambda_{ij}^\alpha |x_i|))$ ,  $\mathbf{x} \in \mathbb{R}^d$ .

From (Embrechts et al., 1997, Lemma A 3.26) we obtain that the df  $F_i$  of  $X_i$  satisfies  $1 - F_i(s) \sim s^{-\alpha} \sum_{j=1}^m \lambda_{ij}^\alpha = s^{-\alpha}$ ,  $1 \leq i \leq d$ , as  $s \rightarrow \infty$  and, thus,

$$\gamma_i = \lim_{s \rightarrow \infty} \frac{1 - F_i(s)}{1 - F_\kappa(s)} = 1, \quad 1 \leq i \leq d,$$

where  $\kappa \in \{1, \dots, d\}$  can be chosen arbitrarily. As a consequence we obtain for the fragility index

$$FI = \frac{\sum_{i=1}^d \gamma_i}{\left\| \sum_{i=1}^d \gamma_i \mathbf{e}_i \right\|_D} = \frac{d}{\sum_{j=1}^m \max_{i \leq d} \lambda_{ij}^\alpha}.$$

*Example 3.3* (GPD-Copula). Take an arbitrary rv  $\mathbf{Z}$  that realizes in  $[0, c]^d$  and which satisfies  $E(Z_i) = 1$ ,  $1 \leq i \leq d$ . Choose  $\beta_1, \dots, \beta_d > 0$  and let  $U$  be a rv, which is uniformly on  $(0, 1)$  distributed and that is independent of  $\mathbf{Z}$ . Put  $\mathbf{X} := (\beta_1 Z_1, \dots, \beta_d Z_d)/U$ . Then  $F_i(x) = P(X_i \leq x) = 1 - \frac{\beta_i}{x}$ ,  $x \geq c\beta_i$ ,  $1 \leq i \leq d$ ,

and the copula of  $\mathbf{X}$  is in the domain of attraction of the EVD  $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_D)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ , with  $\|\mathbf{x}\|_D = E(\max_{1 \leq i \leq d} |x_i| Z_i)$ ,  $\mathbf{x} \in \mathbb{R}^d$ .

Let  $\beta_\kappa = \max_{1 \leq i \leq d} \beta_i$ . Then we have

$$\frac{1 - F_i(s)}{1 - F_\kappa(s)} = \frac{\beta_i}{\beta_\kappa} =: \gamma_i, \quad s \geq c\beta_\kappa, \quad 1 \leq i \leq d,$$

and we obtain for the fragility index corresponding to  $\mathbf{X}$

$$FI = \frac{\sum_{i=1}^d \gamma_i}{E(\max_{1 \leq i \leq d} \gamma_i Z_i)}.$$

Note that the copula  $C$  of  $\mathbf{X}$  is actually a *GPD copula* ((multivariate) generalized Pareto distribution), characterized by the equation  $C(\mathbf{u}) = 1 - \|\mathbf{1} - \mathbf{u}\|_D$  for  $\mathbf{u} \in [0, 1]^d$  close to  $\mathbf{1}$ , see Aulbach et al. (2011). If  $Z_1 = \dots = Z_d$  a.s., then we obtain the maximum-norm  $\|\mathbf{x}\|_D = \max_{1 \leq i \leq d} |x_i|$ , and  $FI = \sum_{i=1}^d \gamma_i / \max_{1 \leq i \leq d} \gamma_i$ .

#### 4. THE EXTENDED FRAGILITY INDEX

The extended FI is the asymptotic expected number of exceedances above a high threshold, conditional on the assumption that there are at least  $m \geq 1$  exceedances:

$$FI(m) := \lim_{s \nearrow} E(N_s \mid N_s \geq m), \quad 1 \leq m \leq d.$$

If the ACDEC corresponding to  $X_1, \dots, X_d$  exists, then, obviously,

$$(4.1) \quad FI(m) = \frac{\sum_{k=m}^d k p_k}{\sum_{k=m}^d p_k}, \quad 1 \leq m \leq d.$$

But now we encounter the problem that we might divide by 0 in (4.1), i.e.,  $\sum_{k=m}^d p_k$  can vanish if  $m \geq 2$ . This is, for example, true for the  $L_1$ -norm. But there are other norms in dimension  $d \geq 3$  such that  $\sum_{k=m}^d p_k = 0$ , see Falk and Tichy (2010). In this section we establish a characterization of  $\sum_{k=m}^d p_k = 0$  also in that case, where the initial  $X_1, \dots, X_d$  follow different distributions.

**Lemma 4.1.** *Assume the conditions of Corollary 2.2 and put  $I := \{i \in \{1, \dots, d\} : \gamma_i = 0\}$ . Then we obtain  $\sum_{k=m}^d p_k = 0$  for  $m > m^* := |I^c| = d - |I|$ .*

*Proof.* Without loss of generality we can assume that  $I \neq \emptyset$ . Recall, moreover, that  $\gamma_\kappa = 1$ , i.e.,  $I \neq \{1, \dots, d\}$  as well. We have

$$a_k = \lim_{s \uparrow \omega^*} \frac{P(N_s = k)}{1 - F_\kappa(s)} = \lim_{s \uparrow \omega^*} \sum_{\substack{S \subset \{1, \dots, d\} \\ |S| = k}} \frac{P(X_i > s, i \in S, X_j \leq s, j \in S^c)}{1 - F_\kappa(s)}.$$

If  $|S| = k \geq m^* + 1$ , then  $S$  must contain an index  $i_S$ , say, with  $i_S \in I$ . We, thus, obtain for  $k \geq m^* + 1$

$$a_k \leq \limsup_{s \uparrow \omega^*} \sum_{\substack{S \subset \{1, \dots, d\} \\ |S| = k}} \frac{P(X_{i_S} > s)}{1 - F_\kappa(s)} = \sum_{\substack{S \subset \{1, \dots, d\} \\ |S| = k}} \lim_{s \uparrow \omega^*} \frac{1 - F_{i_S}(s)}{1 - F_\kappa(s)} = 0.$$

□

The following characterization is the main result of this section. It is formulated in terms of different representations of a multivariate EVD  $G$  on  $\mathbb{R}^d$  with standard

negative exponential margins  $G(x\mathbf{e}_i) = \exp(x)$ ,  $x \leq 0$ ,  $1 \leq i \leq d$ . We have for  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$

$$\begin{aligned} (\text{Hofmann}) \quad G(\mathbf{x}) &= \exp(-\|\mathbf{x}\|_D) \\ (\text{Pickands-de Haan-Resnick}) \quad &= \exp\left(-\int_{S_d} \max(-u_i x_i) \mu(d\mathbf{u})\right) \\ (\text{Balkema-Resnick}) \quad &= \exp\left(-\nu\left([-\infty, \mathbf{x}]^{\mathbb{G}}\right)\right), \end{aligned}$$

where  $\|\cdot\|_D$  is some norm on  $\mathbb{R}^d$  with  $\|\mathbf{e}_i\|_D = 1$ ,  $1 \leq i \leq d$ ,  $\mu$  is the *angular measure* on the unit simplex  $S_d = \{\mathbf{u} \in [0, 1]^d : \sum_{i=1}^d u_i = 1\}$ , satisfying  $\mu(S_d) = d$ ,  $\int_{S_d} u_i \mu(d\mathbf{u}) = 1$ ,  $1 \leq i \leq d$ , and  $\nu$  is the  $\sigma$ -finite *exponent measure* on  $[-\infty, 0]^d \setminus \{\infty\}$ ; for details we refer to Falk et al. (2010). We also include the fact that each  $D$ -norm can be generated by nonnegative and bounded rv  $Z_1, \dots, Z_d$  with  $E(Z_i) = 1$ ,  $1 \leq i \leq d$ , as

$$(4.2) \quad \|\mathbf{x}\|_D = E\left(\max_{1 \leq i \leq d} (|x_i| Z_i)\right), \quad \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

This is a consequence of the Pickands-de Haan-Resnick representation. The rv  $\mathbf{Z} = (Z_1, \dots, Z_d)$  is called *generator* of  $\|\cdot\|_D$ . Note that each rv  $\mathbf{Z} = (Z_1, \dots, Z_d)$  of nonnegative and bounded rv  $Z_i$  with  $E(Z_i) = 1$  generates a  $D$ -norm via equation (4.2).

**Proposition 4.2.** *Assume the conditions of Corollary 2.2 and put  $I = \{i \in \{1, \dots, d\} : \gamma_i = 0\}$ . Then we have  $\sum_{k=m}^d p_k = 0$  for some  $m \leq m^* = |I^{\mathbb{G}}|$  if and only if we have for each subset  $K \subset I^{\mathbb{G}}$  with at least  $m$  elements*

$$\begin{aligned} (4.3) \quad \lim_{s \uparrow \omega^*} \frac{P(X_k > s, k \in K)}{1 - F_\kappa(s)} &= 0 \\ \iff \sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} x_i \mathbf{e}_i \right\|_D &= 0 \quad \text{for all } \mathbf{x} \geq \mathbf{0} \in \mathbb{R}^d \\ \iff \sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \mathbf{e}_i \right\|_D &= 0 \\ (4.4) \quad \iff \min_{k \in K} Z_k &= 0 \quad \text{a.s.} \\ \iff \mu\left(\left\{\mathbf{u} \in S_d : \min_{i \in K} u_i > 0\right\}\right) &= 0 \\ \iff \nu(\times_{k \in K} (-\infty, 0] \times_{i \notin K} [-\infty, 0]) &= 0, \end{aligned}$$

i.e., the projection  $\nu_K := \nu * (\pi_i, i \in K)$  of the exponent measure  $\nu$  onto its components  $i \in K$  is the null measure on  $(-\infty, 0]^{|K|}$ .

While in the (bivariate) case  $K = \{k_1, k_2\}$  the condition

$$\begin{aligned} \sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \mathbf{e}_i \right\|_D &= 0 \\ \iff \|\mathbf{e}_{k_1}\|_D + \|\mathbf{e}_{k_2}\|_D - \|\mathbf{e}_{k_1} + \mathbf{e}_{k_2}\|_D &= 0 \\ \iff \|\mathbf{e}_{k_1} + \mathbf{e}_{k_2}\|_D = 2 = \|\mathbf{e}_{k_1} + \mathbf{e}_{k_2}\|_1 \end{aligned}$$

implies by Takahashi's Theorem (Takahashi (1988)) independence of the marginal distributions  $k_1, k_2$  of the EVD  $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_D)$ ,  $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$ , this is no longer

true for  $|K| \geq 3$ . Take, for example, a rv  $\xi$  that attains only the values 1;2;3 with probability  $1/6$ ;  $1/3$ ;  $1/2$  and put

$$Z_1 := \begin{cases} 0 & \text{if } \xi = 1 \\ \frac{6}{5} & \text{elsewhere} \end{cases}, \quad Z_2 := \begin{cases} 0 & \text{if } \xi = 2 \\ \frac{3}{2} & \text{elsewhere} \end{cases}, \quad Z_3 := \begin{cases} 0 & \text{if } \xi = 3 \\ 2 & \text{elsewhere} \end{cases}.$$

Then  $E(Z_i) = 1$ ,  $i = 1, 2, 3$ ,  $\min_{1 \leq i \leq 3} Z_i = 0$ ,  $E(\max_{1 \leq i \leq 3} Z_i) < 3$  as well as  $E(\max(Z_i, Z_j)) < 2$  for all  $1 \leq i \neq j \leq 3$ , i.e., there is no marginal independence among  $Z_1, Z_2, Z_3$ .

*Proof.* We have by Theorem 2.4 and Lemma 2.3

$$\begin{aligned} \sum_{k=m}^d p_k &= 0 \\ \iff \lim_{s \uparrow \omega^*} \frac{P(N_s \geq m)}{1 - F_\kappa(s)} &= 0 \\ \iff \lim_{s \uparrow \omega^*} \frac{P\left(\bigcup_{\substack{K \subset \{1, \dots, d\} \\ |K| \geq m}} \{X_k > s, k \in K\}\right)}{1 - F_\kappa(s)} &= 0 \\ \iff \lim_{s \uparrow \omega^*} \frac{P(X_k > s, k \in K)}{1 - F_\kappa(s)} &= 0 \text{ for any } K \subset \{1, \dots, d\} \text{ with } |K| \geq m \\ \iff \lim_{s \uparrow \omega^*} \frac{P(X_k > s, k \in K)}{1 - F_\kappa(s)} &= 0 \text{ for any } K \subset I^\mathbb{G} \text{ with } |K| \geq m, \end{aligned}$$

which is equivalence (4.3). Note that  $\sum_{T \subset K} (-1)^{|T|-1} \max_{i \in T} a_i = \min_{k \in K} a_k$  for any set  $\{a_k : k \in K\}$  of real numbers, which can be seen by induction. We, consequently, have

$$\sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \mathbf{e}_i \right\|_D = \sum_{T \subset K} (-1)^{|T|-1} E\left(\max_{i \in T} Z_i\right) = E\left(\min_{i \in T} Z_i\right)$$

and, thus,

$$\sum_{T \subset K} (-1)^{|T|-1} \left\| \sum_{i \in T} \mathbf{e}_i \right\|_D = 0 \iff E\left(\min_{i \in T} Z_i\right) = 0 \iff \min_{k \in K} Z_k = 0 \text{ a.s.}$$

The other equivalences follow from Proposition 5.2 in Falk and Tichy (2010).  $\square$

## 5. EXCEEDANCE CLUSTER LENGTHS

The total number of sequential time points at which a stochastic process exceeds a high threshold is an exceedance cluster length. The mathematical tools developed in the preceding sections enable the computation of its distribution as well. Precisely, denote by  $L_\kappa(s)$  the number of sequential exceedances above the threshold  $s$ , if we have an exceedance at  $\kappa \in \{1, \dots, d\}$ , i.e.

$$L_\kappa(s) := \sum_{k=0}^{d-\kappa} k 1(X_\kappa > s, \dots, X_{\kappa+k} > s, X_{\kappa+k+1} \leq s).$$

We have, in particular,  $L_d(s) = 0 = L_\kappa(s)$ , if  $X_{\kappa+1} \leq s$ . We suppose throughout this section that condition (C) holds for the index  $\kappa \in \{1, \dots, d\}$ . The following auxiliary result will be crucial.



**Lemma 5.1.** *Assume the conditions of Corollary 2.2. Then we obtain for  $\kappa \in \{1, \dots, d\}$  as  $s \nearrow \omega^*$*

$$\begin{aligned} P(L_\kappa(s) \geq k \mid X_\kappa > s) &= P(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s) \\ &= \sum_{\emptyset \neq T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \mathbf{e}_i \right\|_D + o(1) \\ &=: s_\kappa(k) + o(1), \quad 0 \leq k \leq d - \kappa. \end{aligned}$$

*Proof.* From the additivity formula we obtain

$$\begin{aligned} &P(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s) \\ &= \frac{1 - P\left(\bigcup_{0 \leq i \leq k} \{X_{\kappa+i} \leq s\}\right)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \neq T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} P(X_i \leq s, i \in T)}{1 - F_\kappa(s)} \\ &= \frac{1 - \sum_{\emptyset \neq T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} (1 - c \|\sum_{i \in T} \gamma_i \mathbf{e}_i\|_D) + o(1 - F_\kappa(s))}{1 - F_\kappa(s)} \\ &= \sum_{\emptyset \neq T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \mathbf{e}_i \right\|_D + o(1). \end{aligned}$$

□

**Corollary 5.2.** *Suppose in addition to the assumptions in Corollary 2.2 that  $\mathbf{Z}$  is a generator of the  $D$ -norm  $\|\cdot\|_D$ . Then we obtain for  $\kappa \in \{1, \dots, d\}$  as  $s \nearrow \omega^*$*

$$P(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s) = E\left(\min_{\kappa \leq i \leq \kappa+k} (\gamma_i Z_i)\right) + o(1),$$

for  $0 \leq k \leq d - \kappa$ .

Though the distribution of a generator of a  $D$ -norm is not uniquely determined, the preceding result entails that the numbers  $E(\min_{\kappa \leq i \leq \kappa+k} (\gamma_i Z_i))$ ,  $0 \leq k \leq d - \kappa$ , are uniquely determined by the  $D$ -norm.

The asymptotic distribution of the exceedance cluster length, conditional on the assumption that there is an exceedance at time point  $\kappa \in \{1, \dots, d\}$ , is an immediate consequence of Lemma 5.1. It follows from the equation

$$P(L_\kappa(s) = k \mid X_\kappa > s) = P(L_\kappa(s) \geq k \mid X_\kappa > s) - P(L_\kappa(s) \geq k+1 \mid X_\kappa > s).$$

Note, moreover, that  $P(L_\kappa(s) = 0 \mid X_\kappa > s) = 1$  for  $\kappa = d$ .

**Proposition 5.3.** *Assume the conditions of Corollary 2.2. Then we have for  $\kappa < d$  as  $s \nearrow \omega^*$*

$$\begin{aligned} &P(L_\kappa(s) = k \mid X_\kappa > s) \\ &= \begin{cases} \sum_{\emptyset \neq T \subset \{\kappa, \dots, d\}} (-1)^{|T|+1} \left\| \sum_{i \in T} \gamma_i \mathbf{e}_i \right\|_D + o(1), & k = d - \kappa, \\ \sum_{T \subset \{\kappa, \dots, \kappa+k\}} (-1)^{|T|+1} \left\| \gamma_{\kappa+k+1} \mathbf{e}_{\kappa+k+1} + \sum_{i \in T} \gamma_i \mathbf{e}_i \right\|_D + o(1), & 0 \leq k < d - \kappa. \end{cases} \end{aligned}$$

We obtain, for example, for  $\kappa < d$

$$P(L_\kappa(s) = 0 \mid X_\kappa > s) = \|\mathbf{e}_\kappa + \gamma_{\kappa+1} \mathbf{e}_{\kappa+1}\|_D - 1 + o(1),$$

which converges to  $\gamma_{\kappa+1}$  if  $\|\cdot\|_D = \|\cdot\|_1$ . Recall that  $\gamma_\kappa = 1$ .

In terms of a generator  $\mathbf{Z}$  of a  $D$ -norm, Proposition 5.3 becomes the following result.

**Corollary 5.4.** *Assume in addition to the conditions of Corollary 2.2 that  $\mathbf{Z}$  is a generator of the  $D$ -norm  $\|\cdot\|_D$ . Then we have for  $\kappa < d$  as  $s \nearrow \omega^*$*

$$\begin{aligned} & \text{(i) } P(L_\kappa(s) = k \mid X_\kappa > s) \\ &= \begin{cases} E(\min_{\kappa \leq i \leq d}(\gamma_i Z_i)) + o(1), & k = d - \kappa \\ E(\min_{\kappa \leq i \leq \kappa+k}(\gamma_i Z_i) - \min_{\kappa \leq i \leq \kappa+k+1}(\gamma_i Z_i)) + o(1), & 0 \leq k < d - \kappa. \end{cases} \\ & \text{(ii) } P(L_\kappa(s) \leq k \mid X_\kappa > s) \\ &= \begin{cases} 1, & k = d - \kappa \\ 1 - E(\min_{\kappa \leq i \leq \kappa+k+1}(\gamma_i Z_i)) + o(1), & 0 \leq k < d - \kappa. \end{cases} \end{aligned}$$

We, thus, obtain the limit distribution of the exceedance cluster length:

$$\begin{aligned} Q_\kappa([0, k]) &:= \lim_{s \nearrow \omega^*} P(L_\kappa(s) \leq k \mid X_\kappa > s) \\ &= \begin{cases} 1, & k = d - \kappa \\ 1 - E(\min_{\kappa \leq i \leq \kappa+k+1}(\gamma_i Z_i)), & 0 \leq k < d - \kappa. \end{cases} \end{aligned}$$

Take, for example, the generator  $\mathbf{Z} = 2(U_1, \dots, U_d)$ , where the  $U_i$  are independent and uniformly on  $(0, 1)$  distributed rv. If, in addition,  $\gamma_i = 1$ ,  $\kappa \leq i \leq d$ , then we obtain

$$Q_\kappa([0, k]) = \begin{cases} 1, & k = d - \kappa \\ 1 - \frac{2}{k+3}, & 0 \leq k < d - \kappa. \end{cases}$$

Next we compute the asymptotic mean exceedance cluster length.

**Proposition 5.5.** *Assume the conditions of Corollary 2.2 and let  $\mathbf{Z}$  be a generator of the  $D$ -norm  $\|\cdot\|_D$ . Then we have for  $1 \leq \kappa \leq d$*

$$\begin{aligned} E(L_\kappa(s) \mid X_\kappa > s) &= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1) & \text{else} \end{cases} \\ &= \begin{cases} 0, & \text{if } \kappa = d \\ \sum_{k=1}^{d-\kappa} E(\min_{\kappa \leq i \leq \kappa+k}(\gamma_i Z_i)) + o(1) & \text{else.} \end{cases} \end{aligned}$$

*Proof.* Since  $L_\kappa(s)$  attains only nonnegative values, we have for  $\kappa < d$

$$\begin{aligned} E(L_\kappa(s) \mid X_\kappa > s) &= \int_0^\infty P(L_\kappa(s) \geq t \mid X_\kappa > s) dt \\ &= \sum_{k=1}^{d-\kappa} P(L_\kappa(s) \geq k \mid X_\kappa > s) \\ &= \sum_{k=1}^{d-\kappa} P(X_\kappa > s, \dots, X_{\kappa+k} > s \mid X_\kappa > s) \\ &= \sum_{k=1}^{d-\kappa} s_\kappa(k) + o(1). \end{aligned}$$

□

**Corollary 5.6.** *Under the conditions of the preceding result we have for  $\kappa < d$ , if  $\gamma_k > 0$ ,  $1 \leq k \leq d$ ,*

$$\lim_{s \uparrow \omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0$$

*if and only if  $\|x\mathbf{e}_\kappa + y\mathbf{e}_{\kappa+1}\|_D = \|x\mathbf{e}_\kappa + y\mathbf{e}_{\kappa+1}\|_1 = x + y$ ,  $x, y \geq 0$ .*

*Proof.* Note that  $s_\kappa(1) \geq \dots \geq s_\kappa(d - \kappa)$ . We, thus, obtain from Proposition 5.5

$$\lim_{s \uparrow \omega^*} E(L_\kappa(s) \mid X_\kappa > s) = 0 \iff s_\kappa(1) = 0.$$

The assertion is now a consequence of Proposition 6.1 in Falk and Tichy (2010).  $\square$

Suppose in addition to the assumptions of Corollary 2.2 that the components  $X_1, \dots, X_d$  of the rv  $\mathbf{X}$  are exchangeable. Then we have  $\gamma_1 = \dots = \gamma_d = 1$ , as well as  $\left\| \sum_{i \in T} \mathbf{e}_i \right\|_D = \left\| \sum_{i=1}^{|T|} \mathbf{e}_i \right\|_D$  for any nonempty subset  $T \subset \{1, \dots, d\}$ . As a consequence we obtain

$$s_\kappa(k) = \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} \left\| \sum_{i=1}^j \mathbf{e}_i \right\|_D, \quad 0 \leq k \leq d - \kappa,$$

and, thus, by rearranging sums,

$$\begin{aligned} \lim_{s \nearrow} E(L_\kappa(s) \mid X_\kappa > s) &= \sum_{k=1}^{d-\kappa} s_\kappa(k) \\ &= \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \left\| \sum_{i=1}^j \mathbf{e}_i \right\|_D \sum_{k=\max(1, j-1)}^{d-\kappa} \binom{k+1}{j} \\ (5.1) \quad &= -1 + \sum_{j=1}^{d-\kappa+1} (-1)^{j+1} \binom{d-\kappa+2}{j+1} \left\| \sum_{i=1}^j \mathbf{e}_i \right\|_D, \end{aligned}$$

where the final equality follows from the general equation  $\sum_{r=n}^N \binom{r}{n} = \binom{N+1}{n+1}$ .

*Example 5.7* (Marshall-Olkin  $D$ -norm). The Marshall-Olkin  $D$ -norm is the convex combination of the maximum-norm and the  $L_1$ -norm:

$$\|\mathbf{x}\|_{\text{MO}} = \vartheta \|\mathbf{x}\|_1 + (1 - \vartheta) \|\mathbf{x}\|_\infty, \quad \mathbf{x} \in \mathbb{R}^d, \vartheta \in [0, 1],$$

see (Falk et al., 2010, Example 4.3.4). In this case we obtain from equation (5.1)

$$\lim_{s \nearrow} E(L_\kappa(s) \mid X_\kappa > s) = (1 - \vartheta)(d - \kappa),$$

where we have used the general equation  $\sum_{j=0}^m (-1)^j \binom{m}{j} = 0$ .

In the case  $\vartheta = 0$  of complete tail dependence of the margins we, therefore, obtain  $\lim_{s \nearrow} E(L_\kappa(s) \mid X_\kappa > s) = d - \kappa$ , which is the full possible length, whereas in the tail independence case  $\vartheta = 1$  we obtain the shortest length  $\lim_{s \nearrow} E(L_\kappa(s) \mid X_\kappa > s) = 0$ , which is in complete accordance with Corollary 5.6.

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